Superradiance and the Statistical-Mechanical Entropy of Rotating BTZ Black Holes

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Abstract

We have considered the divergence structure in the brick-wall model for the statistical mechanical entropy of a quantum field in thermal equilibrium with a black hole which *rotates*. Especially, the contribution to entropy from superradiant modes is carefully incorporated, leading to a result for this contribution which corrects some previous errors in the literature. It turns out that the previous errors were due to an incorrect quantization of the superradiant modes. Some of main results for the case of rotating BTZ black holes are that the entropy contribution from superradiant modes is positive rather than negative and also has a leading order divergence as that from nonsuperradiant modes. The total entropy, however, can still be identified with the Bekenstein-Hawking entropy of the rotating black hole by introducing a universal brick-wall cutoff. Our correct treatment of superradiant modes in the "angular-momentum modified canonical ensemble" also removes unnecessary introductions of regulating cutoff numbers as well as ill-defined expressions in the literature.

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Since Bekenstein [1] suggested that black holes carry an intrinsic entropy proportional to the surface area of the event horizon, and Hawking [2] provided a physical basis for this idea by considering quantum effect, there have been various approaches to understanding the black hole entropy. One of them is the so-called "brick-wall model" introduced by 't Hooft [3]. He has considered a quantum gas of scalar particles propagating just outside the event horizon of the Schwarzschild black hole. The entropy obtained just by applying the usual statistical mechanical method to this system turns out to be divergent due to the infinite blue shift of waves at the horizon. 't Hooft, however, has shown that the leading order term on the entropy has the same form as the Bekenstein-Hawking formula for the black hole entropy by introducing a brick-wall cutoff which is a property of the horizon only and is the order of the Planck length. The appearance of this divergence [4, 5] and relationships of this "statistical-mechanical" entropy of quantum fields near a black hole with its entanglement entropy [6] and quantum excitations of the black hole [7] have been studied, leading a great deal of interest recently [8].

The brick-wall model originally applied to the four dimensional Schwarzschild black hole [3] has been extended to various situations. The application to the case of rotating black holes has also been done for scalar fields in BTZ black holes in three-dimensions [9, 10] and in Kerr-Newman and other rotating black holes in four-dimensions [11, 12]. In a background spacetime of rotating black holes, it is well known that scalar fields have a special class of mode solutions, giving superradiance. It is claimed in Ref. [10] that the statistical-mechanical entropy of a scalar matter is not proportional to the "area"(*i.e.*, the circumference in the three-dimensional case) of the horizon of a rotating BTZ black hole and that the divergent parts are not necessarily due to the existence of the horizon. Contrary to it, in Ref. [9], the leading divergent term on the entropy is proportional to the "area" of the horizon, and it is possible to introduce a universal brick-wall cutoff which makes the entropy equivalent to the black hole entropy. Moreover, it is claimed in Ref. [9] that the contribution from superradiant modes to entropy is *negative* and its divergence is in a *subleading* order compared to that from nonsuperradiant modes. On the other hand, for the case of Kerr black holes in Ref. [12], the divergence is in the leading order but the entropy contribution is still negative.

One may expect that the leading contribution to the entropy comes from the region very near the horizon as in the case of nonrotating black holes. Since the vicinity of a rotating horizon can also be approximated by the Rindler metric, it is seemingly that the essential feature of the leading contribution will be same as that in nonrotating cases. Our study in detail shows this naive expectation is indeed true. That is, we point out that previous erroneous results appeared in the literature were mainly related to an incorrect quantization of superradiant modes. For the case of rotating BTZ black holes, we have explicitly shown that superradiant modes also give leading order divergence to the entropy

as nonsuperradiant ones. Moreover, its entropy contribution is *positive* rather than negative found in Refs. [9, 12]. However, the total entropy of quantum field can still be identified with the Bekenstein-Hawking entropy by introducing a universal brick-wall cutoff. It also has been shown that the correct quantization of superradiant modes in the calculation of the "angular-momentum modified canonical ensemble" removes various unnecessary regulating cutoff numbers as well as ill-defined expressions in the literature.

Let us consider a quantum gas of scalar particles confined in a box near the horizon of a stationary rotating black hole. The free scalar field satisfies the Klein-Gordon equation given by $(\Box + \mu^2)\phi = 0$ with periodic boundary conditions

$$\phi(r_+ + h) = \phi(L). \tag{1}$$

Here, r_+, $r_+ + h$, and L are the radial coordinates of the horizon and the inner and outer walls of a "spherical" box, respectively. Suppose that this boson gas is in a thermal equilibrium state at temperature β^{-1}. Due to the existence of an ergoregion just outside the event horizon, any thermal system sitting in this region must rotate with respect to an observer at infinity. Accordingly, in order to obtain the appropriate grand canonical ensemble for this rotating thermal system, one should introduce an angular momentum reservoir as well in addition to a heat bath/particle reservoir characterized by temperature $T = \beta^{-1}$ and angular speed Ω with respect to an observer at infinity [13]. All thermodynamic quantities can be derived by the partition function $Z(\beta, \Omega)$ given by

$$Z(\beta, \Omega) = \mathrm{Tr}\, e^{-\beta :(\hat{H} - \Omega \hat{J}):}, \tag{2}$$

where $:\hat{H}:$ and $:\hat{J}:$ are the normal ordered Hamiltonian and angular momentum operators of the quantized field, respectively [14, 13, 8]. Here, we assume that particle number of the system is indefinite.

As usual, by using the single-particle spectrum, one can obtain the free energy $F(\beta, \Omega)$ of the system in the following form

$$\begin{aligned} \beta F &= -\ln Z = -\sum_{\lambda} \ln \sum_{k} [e^{-\beta(\varepsilon_\lambda - \Omega j_\lambda)}]^k \qquad (3) \\ &= \begin{cases} -\sum_\lambda \ln[1 + e^{-\beta(\varepsilon_\lambda - \Omega j_\lambda)}] & \text{for fermions,} \\ \sum_\lambda \ln[1 - e^{-\beta(\varepsilon_\lambda - \Omega j_\lambda)}] & \text{for bosons with } \varepsilon_\lambda - \Omega j_\lambda > 0. \end{cases} \qquad (4) \end{aligned}$$

where λ denotes the single-particle states for the free gas in the system. $\varepsilon_\lambda = \langle 1_\lambda|\, :\hat{H}:\, |1_\lambda\rangle$ and $j_\lambda = \langle 1_\lambda|\, :\hat{J}:\, |1_\lambda\rangle$ are normal ordered energy and angular momentum associated with single-particle states λ, respectively. The occupation number $k = 0, 1, 2, \cdots$ for bosonic fields and $k = 0, 1$ for fermionic fields. Note that, if there exists a bosonic single-particle state with

its energy ε_λ and angular momentum j_λ such that $\varepsilon_\lambda - \Omega j_\lambda < 0$, the expression in Eq. (3) becomes divergent and so is ill-defined. In order to compute the free energy in Eq. (3), one must know all single-particle states and their corresponding values of ε_λ and j_λ for a given system.

As pointed out in Refs. [15, 16], the quantization of matter fields on a stationary *rotating* axisymmetric black hole background is somewhat unusual due to superradiant modes which occur in the presence of an ergoregion. The mode solutions will be of the form, $\phi(x) \sim e^{-i\omega t + im\varphi}$, because this background spacetime possesses two Killing vector fields denoted by ∂_t and ∂_φ. Since the partition function in Eq. (2) is defined with respect to an observer at infinity, the vacuum state to be defined by the standard quantization procedure should be natural to that observer at infinity in the far future. Thus, we expand the neutral scalar field in terms of a complete set of mode solutions as follows:

$$\begin{aligned}\phi(x) &= \sum_m \int_0^\infty d\omega (b^{\rm out}_{\omega m} u^{\rm out}_{\omega m} + b^{\dagger \rm out}_{\omega m} u^{* \rm out}_{\omega m}) + \sum_m \int_{m\Omega_H}^\infty d\omega (b^{\rm in}_{\omega m} u^{\rm in}_{\omega m} + b^{\dagger \rm in}_{\omega m} u^{* \rm in}_{\omega m}) \qquad (5)\\ &\quad + \sum_m \int_0^{m\Omega_H} d\omega (b^{\rm in}_{-\omega -m} u^{\rm in}_{-\omega -m} + b^{\dagger \rm in}_{-\omega -m} u^{* \rm in}_{-\omega -m}). \qquad (6)\end{aligned}$$

Here $u^{\rm out}(x)$ describes unit outgoing flux to the future null infinity $\mathcal{I}^+$ and zero ingoing flux to the horizon $\mathcal{H}^+$ while $u^{\rm in}(x)$ describes unit ingoing flux to $\mathcal{H}^+$ and zero outgoing flux to $\mathcal{I}^+$. These mode solutions are orthonormal

$$\langle u^{\rm out}_{\omega m}\,,\, u^{\rm out}_{\omega' m'}\rangle = \langle u^{\rm in}_{\omega m}\,,\, u^{\rm in}_{\omega' m'}\rangle = \langle u^{\rm in}_{-\omega -m}\,,\, u^{\rm in}_{-\omega' -m'}\rangle = \delta(\omega - \omega')\delta_{mm'} \qquad (7)$$

with respect to the Klein-Gordon inner product

$$\langle \phi_1\,,\, \phi_2\rangle = \frac{i}{2}\int_{t=\rm const.} \phi_1^* \overleftrightarrow{\partial_\mu} \phi_2 \; d\Sigma^\mu. \qquad (8)$$

Note that $u_{\omega m}(x) \sim e^{-i\omega t + im\varphi}$ and we suppressed other quantum numbers. Modes with $\tilde\omega = \omega - \Omega_H m < 0$ exhibit the so-called superradiance. Here Ω_H is the angular speed of the horizon with respect to an observer at infinity. An observer at infinity would measure positive frequency for all modes $u^{\rm out}_{\omega m}$ and $u^{\rm in}_{\omega m}$ with $\tilde\omega > 0$, but measure negative frequency for $u^{\rm in}_{-\omega -m}$ with $\tilde\omega < 0$. A ZAMO [17] near the horizon, however, would see positive frequency waves for $u^{\rm in}_{-\omega -m}$ with $\tilde\omega < 0$ as well as for $u^{\rm in}_{\omega m}$ with $\tilde\omega > 0$. Hence, in the terminology of Ref. [15], we adopt the "distant-observer viewpoint" for $u^{\rm out}(x)$ and the "near-horizon viewpoint" for $u^{\rm in}(x)$. And the conventions are chosen so that they agree with viewpoints.

Now the mode solutions for particles confined in the near-horizon box would be constructed by linearly superpose $u^{\rm in}$ and $u^{\rm out}$ above as follows:

$$\phi_{\omega m}(x) \sim \begin{cases} u^{\rm out}_{\omega m} + \alpha_{\omega m} u^{\rm in}_{\omega m} & \text{for } \tilde\omega > 0, \\ u^{\rm out}_{\omega m} + \alpha_{\omega m} u^{* \rm in}_{-\omega -m} & \text{for } \tilde\omega < 0, \end{cases} \qquad (9)$$

with appropriate normalization factor. $\alpha_{\omega m}$ is chosen so that the modes satisfy the periodic boundary condition in Eq. (1). Thus, only some discrete (real) values of ω will be allowed [19]. $\phi_{\omega m}(x)$ are understood to be cut off everywhere outside the box. Note that $\phi_{\omega m}(x) \sim e^{-i\omega t+im\varphi}$ for all $\tilde{\omega}$.

The inner product of these modes becomes

$$\langle \phi_{\omega m}\,,\,\phi_{\omega' m'}\rangle = \delta_{\omega\omega'}\delta_{mm'}\int_{r_+ + h}^{L}(\omega - \Omega_0 m)|\phi_{\omega m}|^2 N^{-1} d\Sigma, \tag{10}$$

where we have used $d\Sigma^\mu = n^\mu d\Sigma$ and the unit normal to a $t =$ const. surface $n^\mu = N^{-1}(\partial_t + \Omega_0\partial_\varphi)^\mu$. Here $\Omega_0(r)$ is the angular speed of ZAMO's [17]. Since $\Omega_0(r) \le \Omega_H = \Omega_0(r = r_+)$, the norm of a mode solution with $\omega > 0$ is positive if $\tilde{\omega} = \omega - \Omega_H m > 0$. When $\tilde{\omega} < 0$, the norm could be either positive or negative depending on the radial behavior of the solution. If the norm of $\phi_{\omega m}(x)$ is negative, we can easily see that $\phi_{-\omega -m}(x) \sim e^{i\omega t - im\varphi}$ has the positive norm. Let us define a set SR consisting of mode solutions $\phi_{\omega m}$ with $\omega > 0$ whose norms are negative. Then, the quantized field inside the box can be expanded in terms of orthonormal mode solutions as

$$\phi(x) = \sum_{\lambda \notin \mathrm{SR}}[a_{\omega m}\phi_{\omega m}(x) + a^\dagger_{\omega m}\phi^*_{\omega m}(x)] + \sum_{\lambda \in \mathrm{SR}}[a_{-\omega -m}\phi_{-\omega -m}(x) + a^\dagger_{-\omega -m}\phi^*_{-\omega -m}(x)], \tag{11}$$

where the single-particle states are labeled by $\lambda = (\omega, m)$. The Hamiltonian operator in the reference frame of a distant observer at infinity becomes then

$$\begin{aligned} H &= \sum_{\lambda \notin \mathrm{SR}} \omega(a_{\omega m}a^\dagger_{\omega m} + a^\dagger_{\omega m}a_{\omega m}) + \sum_{\lambda \in \mathrm{SR}}(-\omega)(a_{-\omega -m}a^\dagger_{-\omega -m} + a^\dagger_{-\omega -m}a_{-\omega -m}) \\ &= \sum_{\lambda \notin \mathrm{SR}} \omega(N_{\omega m} + \frac{1}{2}) + \sum_{\lambda \in \mathrm{SR}}(-\omega)(N_{-\omega -m} + \frac{1}{2}), \end{aligned} \tag{12}$$

where $N_{\omega m} = a^\dagger_{\omega m}a_{\omega m}$ and $N_{-\omega -m} = a^\dagger_{-\omega -m}a_{-\omega -m}$ are number operators. Now, by following the standard procedure for defining a vacuum state and single-particle states [16, 15], we can easily see that $(\varepsilon_\lambda, j_\lambda) = (\omega, m)$ for single-particle states $\lambda = (\omega, m) \notin$ SR while $(\varepsilon_\lambda, j_\lambda) = (-\omega, -m)$, instead of (ω, m), for single-particle states $\lambda = (\omega, m) \in$ SR. In Refs. [9, 10, 12], however, $(\varepsilon_\lambda, j_\lambda) = (\omega, m)$ for $\lambda \in$ SR have been used, and our study shows that this error comes from the incorrect quantization of superradiant modes. This important difference is a peculiar feature of the quantization of matter fields in the presence of an ergoregion and turns out to make our "angular-momentum modified canonical ensemble" in Eq. (2) being well-defined as shall be shown below in detail. It also makes somewhat unphysical treatment of superradiant modes and introduction of various cutoff numbers unnecessary in the calculation of statistical-mechanical entropy in the literature. For example, in Ref. [10], a

cutoff in the occupation number k was introduced to avoid the divergent sum for superradiant modes in the log in Eq. (3).

In general, ω is discrete due to the finite size of the box, but the gap between adjacent values goes small as the size of the thermal box becomes large. In this continuous limit, one may introduce the density function defined by $g(\omega, m) = \partial n(\omega, m)/\partial\omega$ where $n(\omega, m)$ is the number of mode solutions whose frequency or energy is below ω for a given value of angular momentum m. Thus, $g(\omega, m)d\omega$ represents the number of single-particle states whose energy lies between ω and $\omega + d\omega$, and whose angular momentum is m. Using this density function, the free energy in Eq. (3) can be re-expressed as

$$\beta F = -\sum_m \int d\omega\, g(\omega, m) \ln \sum_k [e^{-\beta(\varepsilon_\lambda - \Omega j_\lambda)}]^k. \tag{13}$$

The angular speed Ω in Eq. (13) is a thermodynamic parameter defined, in principle, by its appearance in the thermodynamic first law for the reservoir, namely $TdS = dE - \Omega dJ + \cdots$. Since a particle cannot move faster than the speed of light, its angular velocity with respect to an observer at rest at infinity should be restricted. The possible maximum and minimum angular speeds are

$$\Omega_\pm(r) = \Omega_0(r) \pm \sqrt{(\partial_t \cdot \partial_\varphi / \partial_\varphi \cdot \partial_\varphi)^2 - \partial_t \cdot \partial_t / \partial_\varphi \cdot \partial_\varphi}, \tag{14}$$

respectively. We see that, as $r \to r_+$, the range of angular velocities a particle can take on narrows down(*i.e.*, $\Omega_\pm(r) \to \Omega_H$), and so the angular speed of particles near the horizon will be Ω_H. For a rotating body in flat spacetime, one knows that all subsystems must rotate uniformly when the body is in a thermal equilibrium state [13]. In fact, it is a part of the thermodynamic zeroth law. In curved spacetimes, the uniform rotation of all subsystems in thermal equilibrium may not be true to hold any more. However, since we will be finally interested in the quantum gas only in the vicinity of the horizon, we shall assume $\Omega = \Omega_H$ below.

Now, one can see that the sum in the log in Eq. (13) is defined well for states belonging to SR since $\varepsilon_\lambda - \Omega j_\lambda = -(\omega - \Omega_H m) > 0$ by the definition of the set SR. For some states with $\omega - \Omega_H m < 0$ not belonging to SR, however, the sum becomes divergent. As will be shown below explicitly, however, the main contribution to the entropy of the system comes from the infinite piling up of waves at the horizon. For such localized solutions near the horizon, the signature of norms in Eq. (10) will be determined by that of $\omega - \Omega_H m$ since $\Omega_0(r) \simeq \Omega_H$ for the range of integration giving dominant contributions. Therefore, we assume that all states with $\omega - \Omega_H m < 0$ belong to the set SR. Then, the free energy in Eq. (13) can be written as $F = F_{\rm NS} + F_{\rm SR}$. Here

$$\beta F_{\rm NS} = \sum_{\lambda \notin {\rm SR}} \int d\omega\, g(\omega, m) \ln[1 - e^{-\beta(\omega - \Omega_H m)}], \tag{15}$$

$$\beta F_{\rm SR} \;=\; \sum_{\lambda\in{\rm SR}}\int d\omega\, g(\omega,m)\ln[1-e^{\beta(\omega-\Omega_H m)}]. \tag{16}$$

In general cases, it is highly nontrivial to compute $g(\omega,m)$ exactly except for some cases in two-dimensional black holes [20]. For suitable conditions, however, one can approximately obtain $g(\omega,m)$ by using the WKB method as in the brick-wall model [3]. For simplicity, let us consider a scalar field in a rotating BTZ black hole in 3-dimensions [21]. For the case of Kerr black holes in 4-dimensions, although the essential result is the same as that in the case of BTZ black holes, it requires some modified formulation basically due to the fact that the geometrical property near the horizon changes along the polar angle [22]. The metric of a rotating BTZ black hole is given by

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\varphi - \Omega_0 dt)^2, \tag{17}$$

where

$$N^2 = r^2/l^2 - M + J^2/4r^2 = (r^2-r_+^2)(r^2-r_-^2)/r^2l^2, \tag{18}$$

and the angular speed of ZAMO's is $\Omega_0 = J/2r^2$. Here $r_\pm$ denote the outer and inner horizons, respectively. Note that $\Omega_H = \Omega_0(r=r_+) = r_-/r_+l$. Then, mode solutions are $\phi_{\omega m}(x) = \phi_{\omega m}(r)e^{-i\omega t+im\varphi}$. Here the radial part $\phi_{\omega m}(r)$ satisfies

$$rN^2\frac{d}{dr}\Big[rN^2\frac{d}{dr}\phi_{\omega m}(r)\Big] + r^2N^4k^2(r;\omega,m)\phi_{\omega m}(r) = 0, \tag{19}$$

where

$$k^2(r;\omega,m) = N^{-4}[(\omega-\Omega_+m)(\omega-\Omega_-m) - \mu^2N^2]. \tag{20}$$

Here $\Omega_\pm(r) = \Omega_0(r) \pm N/r$ for a rotating BTZ black hole in Eq (14).

In the WKB approximation, the discrete value of energy ω in Eq. (19) is related to $n(\omega,m)$ as follows

$$\pi n(\omega,m) = \int_{r_++h}^{L} dr\ \text{“}k\text{”}(r;\omega,m), \tag{21}$$

where “k”$(r;\omega,m)$ is set to be zero if $k^2(r;\omega,m)$ becomes negative for given (ω,m) [3]. Since “k”$(r;\omega,m) \simeq N^{-2}$ and $N(r)\to 0$ as one approaches to the horizon, we can easily see that the dominant contribution in Eq. (21) comes from the integration in the vicinity of the horizon as the inner brick-wall approaches to it(*i.e.*, $h\to 0$).

Now, Eq. (15) becomes

$$\begin{aligned}\beta F_{\rm NS} \;&=\; \sum_{\lambda\notin{\rm SR}}\int d\omega\,\frac{\partial}{\partial\omega}\Big[\frac{1}{\pi}\int_{r_++h}^{L}dr\ \text{“}k\text{”}(r;\omega,m)\Big]\ln[1-e^{-\beta(\omega-\Omega_H m)}]\\ &=\; -\frac{\beta}{\pi}\int_{r_++h}^{L}dr\sum_m\int d\omega\,\frac{k(r;\omega,m)}{e^{\beta(\omega-\Omega_H m)}-1}\\ &\quad +\frac{1}{\pi}\int_{r_++h}^{L}dr\sum_m\, k(r;\omega,m)\ln[1-e^{-\beta(\omega-\Omega_H m)}]\Big|_{\omega_{\rm min}(m)}^{\omega_{\rm max}(m)}\end{aligned} \tag{22}$$

by using the integration by parts in ω. For convenience, one can divide $F_{\rm NS}$ into two parts

$$F_{\rm NS} = F_{\rm NS}^{(m>0)} + F_{\rm NS}^{(m<0)}, \tag{23}$$

where

$$F_{\rm NS}^{(m>0)} = -\frac{1}{\pi}\int_{r_+ + h}^{L} dr\, N^{-2}\int_0^\infty dm \int_{\Omega_+ m}^\infty d\omega\, \frac{\sqrt{(\omega-\Omega_+ m)(\omega-\Omega_- m)}}{e^{\beta(\omega-\Omega_H m)}-1} \tag{24}$$

from states with positive angular momenta and

$$\begin{aligned} F_{\rm NS}^{(m<0)} &= -\frac{1}{\pi}\bigg(\int_{r_+ + h}^{r_{\rm erg}} dr\, N^{-2}\int_{-\infty}^0 dm\int_0^\infty d\omega \\ &\quad + \int_{r_{\rm erg}}^{L} dr\, N^{-2}\int_{-\infty}^0 dm\int_{\Omega_- m}^\infty d\omega\bigg)\frac{\sqrt{(\omega-\Omega_+ m)(\omega-\Omega_- m)}}{e^{\beta(\omega-\Omega_H m)}-1} \\ &\quad -\frac{1}{\pi\beta}\int_{r_+ + h}^{r_{\rm erg}} dr\, N^{-2}\int_{-\infty}^0 dm\, \sqrt{\Omega_+\Omega_- m^2}\ln(1-e^{\beta\Omega_H m}) \end{aligned} \tag{25}$$

from states with negative angular momenta. $r_{\rm erg} = \sqrt{M}l$ is the radius of the outer boundary of the ergoregion where $\Omega_-(r = r_{\rm erg}) = 0$. Here we considered a massless scalar field for simplicity. Similarly, from states belonging to SR, Eq. (16) becomes

$$\begin{aligned} F_{\rm SR} &= -\frac{1}{\pi}\int_{r_+ + h}^{r_{\rm erg}} dr\, N^{-2}\int_0^\infty dm\int_0^{\Omega_- m} d\omega\, \frac{\sqrt{(\omega-\Omega_+ m)(\omega-\Omega_- m)}}{e^{-\beta(\omega-\Omega_H m)}-1} \\ &\quad +\frac{1}{\pi\beta}\int_{r_+ + h}^{r_{\rm erg}} dr\, N^{-2}\int_0^\infty dm\sqrt{\Omega_+\Omega_- m^2}\,\ln(1-e^{-\beta\Omega_H m}). \end{aligned} \tag{26}$$

Note that $g = -\partial n/\partial\omega$ for $\lambda \in$ SR, and that the boundary term in $F_{\rm SR}$ exactly cancels that in $F_{\rm NS}^{(m<0)}$.

From Eqs. (24-26), we can obtain leading order dependence on the brick-wall cutoff h for the free energy as follows;

$$\begin{aligned} F_{\rm NS}^{(m>0)} &= -\frac{\zeta(3)}{\beta^3}\frac{r_+^2 l^3}{(r_+^2-r_-^2)^2}\left[\frac{\sqrt{r_+^2-r_-^2}}{2\sqrt{2}}\sqrt{\frac{r_+}{h}} + \frac{r_-}{\pi}\ln(\frac{r_+}{h}) + \vartheta(\sqrt{h})\right], \\ F_{\rm NS}^{(m<0)} &= -\frac{\zeta(3)}{\beta^3}\frac{r_+^2 l^3}{(r_+^2-r_-^2)^2}\left[\frac{r_+^2-r_-^2}{2\pi r_-}\ln(\frac{r_+}{h}) + \vartheta(\sqrt{h})\right], \\ F_{\rm SR} &= -\frac{\zeta(3)}{\beta^3}\frac{r_+^2 l^3}{(r_+^2-r_-^2)^2}\bigg[\frac{\sqrt{r_+^2-r_-^2}}{2\sqrt{2}}\sqrt{\frac{r_+}{h}} - \frac{r_-}{\pi}\ln(\frac{r_+}{h}) - \frac{r_+^2-r_-^2}{2\pi r_-}\ln(\frac{r_+}{h}) \\ &\quad +\vartheta(\sqrt{h})\bigg]. \end{aligned} \tag{27}$$

The entropy of this boson gas which is assumed to be in thermal equilibrium with the rotating black hole can be obtained from the free energy by using the thermodynamic relation, $S = \beta^2 \partial F/\partial\beta|_{\beta=\beta_H} = -3\beta F|_{\beta=\beta_H}$;

$$\begin{aligned} S_{\rm NS} &= \frac{3\zeta(3)}{4\pi^2 l}\left[\frac{\sqrt{r_+^2-r_-^2}}{2\sqrt{2}}\sqrt{\frac{r_+}{h}} + \frac{r_-}{\pi}\ln(\frac{r_+}{h}) + \frac{r_+^2-r_-^2}{2\pi r_-}\ln(\frac{r_+}{h}) + \vartheta(\sqrt{h})\right], \\ S_{\rm SR} &= \frac{3\zeta(3)}{4\pi^2 l}\left[\frac{\sqrt{r_+^2-r_-^2}}{2\sqrt{2}}\sqrt{\frac{r_+}{h}} - \frac{r_-}{\pi}\ln(\frac{r_+}{h}) - \frac{r_+^2-r_-^2}{2\pi r_-}\ln(\frac{r_+}{h}) + \vartheta(\sqrt{h})\right], \end{aligned} \tag{28}$$

where the temperature of a rotating BTZ black hole [21] is

$$\beta_H^{-1} = (r_+^2 - r_-^2)/2\pi r_+ l^2. \tag{29}$$

Now the total entropy of the system becomes

$$S = \frac{3\zeta(3)}{4\sqrt{2}\pi^2}\frac{\sqrt{r_+^2-r_-^2}}{l}\sqrt{\frac{r_+}{h}} + \vartheta(\sqrt{h}). \tag{30}$$

In Ref. [9], it is claimed that the contribution from superradiant modes is a subleading order compared with that from nonsuperradiant modes. In our results above, however, we find that superradiant modes also give a leading order contribution which is in fact exactly same as that from nonsuperradiant modes in the leading order of $\sqrt{r_+/h}$. It should be pointed out that the entropy associated with superradiant modes is *positive* in our result whereas it is *negative* in Refs. [9, 12]. In addition, since the log terms in Eq. (28) are exactly cancelled, our result for the entropy of quantum field smoothly reproduces the correct result in the non-rotating limit(*i.e.*, $J \to 0$ or $r_- \to 0$) whereas the entropy obtained in Refs. [9, 12] becomes divergent in that limit.

If we rewrite the entropy in terms of the brick-wall cutoff in proper length defined as $\bar{h} = \int_{r_+}^{r_++h}\sqrt{g_{\rm rr}}dr$, Eq. (30) becomes

$$S = \frac{3\zeta(3)/8\pi^3}{\bar{h}}\mathcal{C} + \vartheta(\bar{h}), \tag{31}$$

where $\mathcal{C} = 2\pi r_+$ is the circumference of the horizon. Thus, by recovering the dimension and introducing an appropriate brick-wall cutoff

$$\bar{h} = \frac{3\zeta(3)}{16\pi^3} l_P \simeq 7.3 \times 10^{-3} l_P \tag{32}$$

which is a universal constant, one can make the entropy of quantum field finite and being equivalent to the Bekenstein-Hawking entropy of a rotating BTZ black hole [21]

$$S = 4\pi r_+/l_P = S_{\rm BH} \tag{33}$$

in leading order. Here l_P is the Planck length. For a fermionic field, although modes with $\tilde{\omega} < 0$ do not reveal superradiance, it turns out that only the overall numerical factor in Eq. (30) is different. As mentioned before, the extension of our study to the case of Kerr black holes in four-dimensions is straightforward, but requires some modifications mainly due to the polar angle dependence of the near horizon geometry. A calculation in the phase space shows that the essential feature of the leading order divergence in the entropy of quantum fields is same as that of the present case [22].

Other thermodynamic quantities of quantum field such as the angular momentum and internal energy can also be obtained as follows;

$$J_{\rm matter} = -\frac{\partial F}{\partial \Omega}\bigg|_{\beta=\beta_H,\Omega=\Omega_H} = \frac{3\zeta(3)/16\pi^3}{\bar{h}}\,\frac{2r_+r_-}{l} + \vartheta(\ln \bar{h}). \tag{34}$$

Here the derivative with respect to Ω has been taken for Eqs. (24-26). If we put the cutoff value in Eq. (32), we have

$$J_{\rm matter} = \frac{2r_+r_-}{l} = J_{\rm BH}. \tag{35}$$

The internal energy of the system with respect to an observer at infinity is

$$\begin{aligned} E &= \frac{\partial}{\partial\beta}(\beta F)\bigg|_{\beta=\beta_H,\Omega=\Omega_H} + \Omega_H J_{\rm matter} \\ &= \frac{3\zeta(3)/16\pi^3}{\bar{h}}\,\frac{4}{3}\frac{r_+^2+\frac{1}{2}r_-^2}{l^2} + \vartheta(\ln \bar{h}) = \frac{4}{3}M_{\rm BH} - \frac{2}{3}\frac{r_-^2}{l^2}, \end{aligned} \tag{36}$$

where the black hole mass is $M_{\rm BH} = M = (r_+^2 + r_-^2)/l^2$. One can easily see that $J_{\rm matter} \to 0$ and $E \to \frac{4}{3}M_{\rm BH}$ in the limit of non-rotating black holes (*e.g.*, $J_{\rm BH} = J \to 0$). Therefore, we find that the entropy and angular momentum of quantum field can be identified with those of the rotating black hole by introducing a universal brick-wall cutoff although the internal energy is not proportional to the black hole mass.

What kind of relationships could be held among parameters characterizing a rotating black hole and thermodynamic quantities of the system of quantum fields in equilibrium with the black hole? To see this, let us consider a system whose free energy depends on the temperature such that $F(\beta,\Omega,M,J) = \beta^{-3}f(\Omega,M,J)$ [23]. Suppose the entropy of the system is identified with that of the black hole after an appropriate regularization,

$$S = \beta^2\left(\frac{\partial F}{\partial\beta}\right)_\Omega\bigg|_{\beta=\beta_H,\Omega=\Omega_H} = S_{\rm BH} = 4\pi r_+. \tag{37}$$

The internal energy of the system with respect to a "corotating" observer is

$$E' = \left[\frac{\partial}{\partial\beta}(\beta F)\right]_\Omega\bigg|_{\beta=\beta_H,\Omega=\Omega_H} = \frac{2}{3}\frac{S}{\beta_H} = \frac{4}{3}\frac{r_+^2-r_-^2}{l^2}. \tag{38}$$

Now suppose the angular momentum of the system is proportional to that of the rotating black hole,

$$J_{\rm matter} = -\left(\frac{\partial F}{\partial \Omega}\right)_\beta \Big|_{\beta=\beta_H,\Omega=\Omega_H} = \alpha J = \alpha \frac{2r_+ r_-}{l}. \tag{39}$$

The internal energy of the system with respect to an observer at infinity is then

$$E = E' + \Omega_H J_{\rm matter} = \frac{4}{3}\frac{r_+^2 + (3\alpha/2 - 1)r_-^2}{l^2}, \tag{40}$$

which becomes proportional to the mass of the black hole, $M = (r_+^2 + r_-^2)/l^2$, only if $\alpha = 4/3$. Therefore, we expect the relationships are probably

$$J_{\rm matter} = \frac{4}{3}J, \qquad E = \frac{4}{3}M. \tag{41}$$

If we apply the same argument to the case of Kerr black holes in 4-dimensions, we obtain

$$J_{\rm matter} = \frac{3}{4}J, \qquad E = \frac{3}{8}M \tag{42}$$

of which the second relationship has been explicitly shown for the Schwarzschild black hole in the brick-wall model by 't Hooft [3].

We have not obtained the relationships in Eq. (41) at the present letter. The reason for these discrepancies is not understood at the present. It will be very interesting to see how the Pauli-Villars regularization method, which does not require the presence of a brick-wall as shown in Ref. [5] for the case of a charged non-rotating black hole, works for the case of rotating black holes.

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